MATH 2850: EXACT EQUATIONS

RECALL: Given a function of two variables $z = F(x, y)$, the **total differential** is $dz = F_x(x, y)\,dx + F_y(x, y)\,dy$.

EXAMPLE: For $z = x^2 e^{xy} - 2xy^3$, find an expression for dz.

EXAMPLE: Find a family of integral curves for the DE: $\left(2xe^{xy} + x^2 ye^{xy} - 2y^3\right)\,dx + \left(x^3 e^{xy} - 6xy^2\right)\,dy = 0$

DEFINITION: A DE is said to be **exact** if it can be written in the form $F_x(x, y)\,dx + F_y(x, y)\,dy = 0$.

An implicit one-parameter family of integral curves for $F_x(x, y)\,dx + F_y(x, y)\,dy = 0$ is $F(x, y) = C$.

EXACTNESS CRITERIA: If M and N have continuous first partials on an open rectangle R, then the DE

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is exact on R if and only if $M_y(x, y) = N_x(x, y)$.

In this case, there is a function F with $F_x = M$ and $F_y = N$ so that:

$$M(x, y)\,dx + N(x, y)\,dy = 0 \iff F_x(x, y)\,dx + F_y(x, y)\,dy = 0 \iff d[F(x, y)] = 0.$$

Hence, a one parameter family of implicit solutions to $M(x, y)\,dx + N(x, y)\,dy = 0$ is $F(x, y) = C$.

EXAMPLE: Consider the DE: $(2x + ye^{xy})\ dx + (xe^{xy} - \cos(y))\ dy = 0$.

- Show this equation is exact.

- Find a one parameter family of implicit solutions to this DE.

Ans: $x^2 + e^{xy} - \sin(y) = C$.

EXAMPLE: Consider the DE: $\left(2x\sin(xy) + x^2y\cos(xy)\right)\,dx + \left(x^3\cos(xy) - 3y^2\right)\,dy = 0$.

- Show this equation is exact.

- Find a one parameter family of implicit solutions to this DE.

Ans: $x^2\sin(xy) - y^3 = C$.

EXAMPLE: Solve the IVP: $y' = \dfrac{y^2 - 3x^2y}{x^3 - 2xy}$, $y(2) = 1$.

Ans: $x^3y - xy^2 = 6$

HOMEWORK: pg. 79: 1-21 odd, 29, 42-44*

MATH 2850: INTEGRATING FACTORS (REVISITED)

DEFINITION: Given a DE $M(x,y)\,dx + N(x,y)\,dy = 0$, an integrating factor is a function $\mu(x,y)$ so that:

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y) = 0$$

is exact. (If the DE is exact already, then $\mu(x,y) = 1$ is an integrating factor.)

We can try for special cases when $\mu(x,y) = \mu(x)$ or $\mu(x,y) = \mu(y)$ - that is, μ depends only on one variable.

THEOREM: Given a DE $M(x,y)\,dx + N(x,y)\,dy = 0$:

- If $\dfrac{M_y(x,y) - N_x(x,y)}{N(x,y)}$ depends on x only, $\mu(x) = e^{\int \frac{M_y(x,y) - N_x(x,y)}{N(x,y)}\,dx}$ is an integrating factor.
- If $\dfrac{N_x(x,y) - M_y(x,y)}{M(x,y)}$ depends on y only, $\mu(y) = e^{\int \frac{N_x(x,y) - M_y(x,y)}{M(x,y)}\,dy}$ is an integrating factor.

DERIVATION:

EXAMPLE: Consider the DE: $\left(1 - y^3\right)\, dx + xy^2\, dy = 0$.

- Show the DE is not exact.

- Find an integrating factor of the form $\mu(x)$.

Ans: $\mu(x) = x^{-4}$

- Find an implicit family of solutions to this DE using your integrating factor.

Ans: $C\,x^3 + y^3 = 1$

NOTE: This DE is also separable and it is also linear in the variable x.

EXAMPLE: Consider the IVP: $(y^4 - y)\,dx - x\,dy = 0,\ y(2) = -1.$

- Show the DE is not exact.

- Find an integrating factor of the form $\mu(y)$.

- Find a solution to the IVP.

Ans: $\dfrac{xy}{\sqrt[3]{y^3-1}} = \sqrt[3]{4}$

NOTE: This DE is also separable and it is also linear in the variable x.

HOMEWORK: pg. 91: 1-15 odd